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Quantum-classical correspondence and nonclassical states generation in dissipative quantum optical systems

Kirill N. Alekseev[1,2*], Natasha V. Alekseeva[1], and Jan Peřina[2**]

[1] *Kirensky Institute of Physics, Russian Academy of Sciences, Krasnoyarsk 660036, Russia*

[2] *Department of Optics and Joint Laboratory of Optics*

Palacký University, 17. listopadu 50, 772 07 Olomouc, Czech Republic

We develop a semiclassical method for the determination of the nonlinear dynamics of dissipative quantum optical systems in the limit of large number of photons N, based on the $1/N$-expansion and the quantum-classical correspondence. The method has been used to tackle two problems: to study the dynamics of nonclassical state generation in higher-order anharmonic dissipative oscillators and to establish the difference between the quantum and classical dynamics of the second-harmonic generation in a self-pulsing regime. In addressing the first problem, we have obtained an explicit time dependence of the squeezing and the Fano factor for an arbitrary degree of anharmonism in the short-time approximation. For the second problem, we have established analytically a characteristic time scale when the quantum dynamics differs insignificantly from the classical one.

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I. INTRODUCTION

The situation when nonlinear interactions involve a large number of photons, N, is quite typical of many of the problems of quantum and nonlinear optics [1–3]. Hiedmann *et al.* suggested [4] to use the method of the $1/N$-expansion [5] to describe the nonlinear dynamics of the mean values and second-order cumulants of a quantum system in the limit $N \gg 1$. Following the general scheme of that method [5], an exact or approximate solution can be found first in terms of the coherent state representation in the classical limit $N \to \infty$ and then adjusted by adding the quantum corrections. The method proves to be particularly convenient when the generation dynamics of nonclassical states needs to be determined [4]. We have recently developed the method further to study the enhanced squeezing at the transition to quantum chaos [6–8].

Papers [4,6,7] are concerned with consideration of the problems of nondissipative quantum systems only. In this paper we extend the method to dissipative quantum systems. For quantum systems without dissipation, the lowest-order of the $1/N$-expansion is equivalent to the linearization in terms of the classical solution [6,7], whereas in dissipative systems, as will be demonstrated herein, the solution of motion equations for variations near the classical trajectory cannot provide complete information on the dynamics of quantum fluctuations even in the lowest order of $1/N$. We will show that the influence of reservoir on the dynamics of the expectation values and dispersions, which is different from the energy dissipation, always exists; it has the quantum nature and can not be neglected even in the semiclassical limit. Specific manifestations of the effect will however depend on the type of the attractor in the underlying classical dynamic system. For systems with a simple attractor in the classical limit, the "quantum diffusion" associated with the reservoir quantum fluctuations will not introduce any new physical effects in the dynamics of the main system, at least in the short-time limit. In contrast to that, for a stable limit cycle, such a diffusion appears to be the main mechanism responsible for the difference between the classical and quantum dynamics at $N \gg 1$.

Along with the presentation of a general formalism, two typical examples of quantum optical systems with a simple attractor and a stable limit cycle in the classical limit $N \to \infty$ will be considered: the dissipative higher-order anharmonic oscillator and the self-pulsing regime of intracavity second-harmonic generation (SHG). We will show how the method of $1/N$-expansion can be used to investigate the dynamics of nonclassical state generation and to determine a time scale for a correct classical description of the dissipative quantum dynamics.

A quantum anharmonic oscillator with the Kerr-type nonlinearity is one of the simplest and most popular models employed in the description of quantum statistical properties of the light interacting with a nonlinear medium [1,9]. The Kerr oscillator model with a third-order nonlinearity yields an exact solution in both nondissipative [10] and dissipative limits [9]. However, due to the complexity of the solution in the dissipative case, numerical methods or special approximate analytical methods have to be used to determine statistical properties of the radiation in a most relevant experimental case involving a large number of photons. Moreover, there are no exact solutions available for the model of the anharmonic oscillator with a higher-order nonlinearity.

In this paper, a simple and explicit time dependencies of the degree of squeezing and the Fano factor are obtained

analytically in the model of anharmonic oscillator of an arbitrary order for the most interesting experimental situation featuring higher intensities ($N \gg 1$) and short-time interactions. As another example of application of the $1/N$-expansion, we consider the self-pulsing in SHG [11]. Such an oscillatory regime, corresponding to the limit cycle, was observed experimentally in [12]. There are quite a few papers dealing with the development of approximate analytical and numerical methods with the purpose of describing different dynamic regimes in SHG in terms of quantum mechanics [13–17]. In particular, Savage [14] calculated the Q distribution function in the Gaussian approximation about the classical limit cycle. He demonstrated numerically that in the classical limit, the initial rapid collapse of the Q distribution in the neighbourhood of the limit cycle was followed by the diffusion around the limit cycle. However, the author did not offer any analytical solution of the problem or explanation of the physics of the effect observed.

In this paper we show that the diffusion around the classical limit cycle can be obtained as a solution of the motion equations for low-order cumulants by using the $1/N$-expansion technique. This enables us to find the time scale $t \ll t^*$ with $t^* \simeq 2N\gamma^{-1}$ (γ is a damping constant), for a correct classical description of self-oscillations in SHG. The resultant estimate is consistent with that obtained for t^* numerically in [14]. Finally, we interpret the quantum diffusion around the limit cycle as having been caused by the effect of the reservoir vacuum on the SHG dynamics.

The structure of the paper is as follows. Section II describes a general formalism of the $1/N$-expansion applicable to an arbitrary single-mode quantum dissipative system and presents the solution of the motion equations for mean values and second-order cumulants obtained in the first order of $1/N$. Sections III and IV deal with the nonclassical state generation dynamics in higher-order anharmonic oscillators and the quantum-classical correspondence for the self-pulsing regime in SHG, respectively. The final section contains a summary and concluding remarks.

II. $1/N$-EXPANSION AND QUANTUM-CLASSICAL CORRESPONDENCE

First of all, we need to generalize the approach of [7] for the case of systems with dissipation. As an illustrative example we consider a quantum anharmonic oscillator with the Hamiltonian in the interaction picture

$$H = \Delta b^\dagger b + \frac{\lambda_l}{l+1}\left(b^\dagger b\right)^{l+1}, \quad [b, b^\dagger] = 1, \tag{1}$$

where the operators b and $b^\dagger$ describe a single mode of a quantum field and the constant λ_l is proportional to a $(2l+1)$-order nonlinear susceptibility of a nonlinear medium (l is an integer), Δ is the light frequency detuning from the characteristic frequency of quantum transition, and $\hbar \equiv 1$. Everywhere in this paper we use the normal ordering of operators. The oscillator interacts with an infinite linear reservoir of finite temperature. The Hamiltonians of the reservoir and interaction of the oscillator with reservoir are defined as follows

$$H_{\mathrm{r}} = \sum_j \psi_j (d_j^\dagger d_j + 1/2), \quad H_{\mathrm{int}} = \sum_j \left(\kappa_j d_j b^\dagger + \mathrm{H.c.}\right), \tag{2}$$

where the Bose operator d_j ($[d_j, d_k^\dagger] = \delta_{jk}$) describes an infinite reservoir with characteristic frequencies ψ_j, and κ_j are the coupling constants between reservoir modes and the oscillator. Introduce new scaled operators $a = b/N^{1/2}$, $c_j = d_j/N^{1/2}$ and the Hermitian conjugates satisfying the commutation relations

$$[a, a^\dagger] = 1/N, \quad [c_j, c_k^\dagger] = \delta_{jk}/N. \tag{3}$$

In the classical limit $N \to \infty$, we have commuting classical c-numbers instead of operators. Now the full Hamiltonian $H = H_0 + H_r + H_{int}$ may be rewritten as $H = N\mathcal{H}$, where $\mathcal{H}$ has the same form as (1) and (2) but for the following replacements

$$b \to a, \quad b^\dagger \to a^\dagger, \quad d_j \to c_j, \quad d_j^\dagger \to c_j^\dagger, \quad \text{and} \quad \lambda_l \to g_l(N) \equiv \lambda N^l. \tag{4}$$

It can be shown that the photon-number dependent constant $g_l(N)$ provides a correct time scale of oscillations for the nonlinear oscillator (1) in the classical limit (for the case of Kerr nonlinearity with $l = 1$, see, *e. g.* [18]). Note that $\mathcal{H}$ can have an explicit time dependence in the general case [7]. Within a standard Heisenberg-Langevin approach, the equation of motion has the form ([1], chap. 7)

$$\dot{a} = -i\left(\Delta - i\frac{\gamma}{2}\right)a + V + L(t), \tag{5}$$

where $V = \partial\mathcal{H}_0/\partial a^\dagger$, $\gamma = 2\pi|\kappa(\omega)|^2\rho(\omega)$ is the damping constant, $\rho(\omega)$ being the density function of reservoir oscillators, which spectrum is considered to be flat. The Langevin force operator $L(t)$ is in a standard relation to the operators $\{c_j\}$ of the reservoir [1]. The properties of $L(t)$ [1] in our notations (4) may be rewritten as

$$\langle L(t)\rangle_R = \langle L^\dagger(t)\rangle_R = 0, \quad \langle L^\dagger a\rangle_R + \langle a^\dagger L\rangle_R = \gamma\frac{\langle n_d\rangle}{N}, \quad \langle La\rangle_R + \langle aL\rangle_R = 0, \tag{6}$$

where the averaging is performed over the reservoir variables and $\langle n_d\rangle$ is a single-mode mean number of reservoir quanta (phonons), that is related to temperature T as $\langle n_d\rangle = \left[\exp\left(\frac{\omega}{kT}\right) - 1\right]^{-1}$, where k is the Boltzmann constant and ω is the characteristic phonon frequency. From the Heisenberg-Langevin equations for a, a^2 and the Hermitian conjugated equations, by using Eqs. (5) and (6), we obtain:

$$\begin{gathered} i\frac{d}{dt}\langle\alpha\rangle = \langle V\rangle - i\frac{\gamma}{2}\langle\alpha\rangle, \\ i\frac{d}{dt}\langle(\delta\alpha)^2\rangle = 2\langle V\delta\alpha\rangle + \langle W\rangle - i\gamma\langle(\delta\alpha)^2\rangle, \\ i\frac{d}{dt}\langle\delta\alpha^*\delta\alpha\rangle = -\langle V^*\delta\alpha\rangle + \langle\delta\alpha^* V\rangle - i\gamma\langle\delta\alpha^*\delta\alpha\rangle + i\gamma\frac{\langle n_d\rangle}{N}, \end{gathered} \tag{7}$$

where $W = (1/N)\partial V/\partial a^\dagger$, $z \equiv \langle a\rangle$, $\langle(\delta\alpha)^2\rangle = \langle a^2\rangle - z^2$, $\langle\delta\alpha^*\delta\alpha\rangle = \langle a^\dagger a\rangle - |z|^2$, and the averaging is performed over both the reservoir variables and the coherent state $|\alpha\rangle = \exp(N\alpha a^\dagger - N\alpha^* a)|0\rangle$ corresponding to the mean photon number $\simeq N$. In deriving Eq. (7), we neglect the insignificant additional detuning introduced to Δ by the interaction with the reservoir [1]. In the absence of no damping $\gamma = 0$, our equations for the mean values and the second-order cumulants (7) are reduced to the corresponding equations in [4,7].

Set of equations (7) is not closed and is basically equivalent to the infinite dynamical hierarchy system for the cumulants of a different order. To truncate it up to the second-order cumulants, we make the substitution $a \to z + \delta\alpha$, where at least initially the mean $z \simeq 1$ and the quantum correction $|\delta\alpha(t=0)| \simeq N^{-1/2} \ll 1$. Using the Taylor expansion of the functions V and W and after some algebra analogous to that used in [7], we get from (7) in the first order of $1/N$ the following self-consistent system of equations for the mean value and the second order cumulants (for details see [19])

$$i\dot{z} = -i\frac{\gamma}{2}z + \langle V\rangle_z + \frac{1}{N}Q(z, z^*, C, C^*, B), \tag{8a}$$

$$i\dot{C} = 2\left(\frac{\partial V}{\partial\alpha}\right)_z C + 2\left(\frac{\partial V}{\partial\alpha^*}\right)_z B - i\gamma C, \tag{8b}$$

$$i\dot{B} = -\left(\frac{\partial V^*}{\partial\alpha}\right)_z C + \left(\frac{\partial V}{\partial\alpha^*}\right)_z C^* - i\gamma\left(B - B^{(0)}\right) \tag{8c}$$

and the corresponding equation for $C^*(t)$ that could be obtained from equation (8b) by way of complex conjugation. The quantum correction to the classical motion Q in Eq. (8a) has the following form

$$Q = \frac{1}{2}\left(\frac{\partial^2 V}{\partial\alpha^2}\right)_z C + \frac{1}{2}\left(\frac{\partial^2 V}{\partial\alpha^{*2}}\right)_z C^* + \left(\frac{\partial^2 V}{\partial\alpha^*\partial\alpha}\right)_z \left(B - \frac{1}{2}\right). \tag{9}$$

In Eqs. (8) and (9) the subscript z means that the values of V and its derivatives are calculated for the mean value z and we have introduced

$$B = N\langle\delta\alpha^*\delta\alpha\rangle + 1/2, \quad C = N\langle(\delta\alpha)^2\rangle. \tag{10}$$

The initial conditions for system (8) are

$$B(0) = 1/2, \quad C(0) = 0, \tag{11}$$

and an arbitrary $z(0) \equiv z_0$ which is of order unity. The equilibrium value of cumulant B in equation (8c) is determined by the mean number of reservoir's quanta and its zero-point energy as

$$B^{(0)} = \langle n_d \rangle + 1/2. \tag{12}$$

Note that the zero-point energy of the reservoir appears in the equations of motion for the cumulants though it was not presented in the Heisenberg equations of motion and even may be dropped from the Hamiltonian redefining a zero of energy. Such "reappearance" of a zero-point field energy is quite common in other problems of quantum theory where a vacuum is responsible for the physical effects [20].

The motion equations for second-order cumulants B and C [Eqs. (8b), (8c)] are linear inhomogeneous equations. Their solution consists of two parts: a general solution of the homogeneous set of equations (*i.e.* without term $+i\gamma B^{(0)}$ in Eq. (8c)) that we denote as $\left(\overline{B}(t), \overline{C}(t)\right)$, and the particular solution of the inhomogeneous equations

$$(B(t), C(t)) = \left(\overline{B}(t), \overline{C}(t)\right) + \left(\gamma B^{(0)} t, 0\right). \tag{13}$$

To find $\left(\overline{B}(t), \overline{C}(t)\right)$ we use the perturbation theory for $N \gg 1$ and as a first step neglect the quantum correction Q/N in Eq. (8a). It is easy to see that the homogeneous equations of motion for cumulants (8b) and (8c) can be obtained from the classical equation (*i.e.* from (8a) with $Q/N \to 0$) by linearization around z (substitution $z \to z + \delta z$, $|\delta z| \ll |z|$), if one writes the dynamical equations for the variables $(\delta z)^2$ and $|\delta z|^2$. The only difference between the linearization of classical motion equations and equations for quantum cumulants (8b), (8c) lies in the impossibility to get the initial conditions (11) for C and B from only initial conditions for linearized classical equations of motion (see also the discussion of this problem in [7]). Hence, we first need to know the classical solution $z_{\rm cl}(t)$, find differentials $dz_{\rm cl}$ and $dz^*_{\rm cl}$, and then use the substitution $\left(\overline{B}(t), \overline{C}(t)\right) \to \left(|dz|^2, (dz)^2\right)$.

Thus, it has become apparent that assuming the actual field deviates little from the coherent state and treating the small deviation as as a first-order correction would not be equivalent to direct linearization around a classical trajectory. Even in the limit $N \to \infty$, we will always deal with the influence of reservoir on the dynamics of the quantum system via the second-order cumulant B, which has the form of quantum diffusion

$$B(t) = \overline{B}(t) + (\langle n_d \rangle + 1/2)\gamma t, \tag{14}$$

where $\overline{B}$ is obtained from linearization around a large mean field. In particular, as follows from Eq. (14), the quantum diffusion also exists for the case of a quiet reservoir $\langle n_d \rangle = 0$.

We now discuss the range of validity of the $1/N$-expansion and the role of quantum diffusion in different classical dynamical regimes. The criterion of validity of the $1/N$-expansion may be represented in two forms. First, the $1/N$-expansion works well, provided the difference between the classical and quantum solutions is small

$$\left| \frac{z(t) - z_{\rm cl}(t)}{z_{\rm cl}(t)} \right| \simeq \frac{1}{N} \frac{\left| \int^t Q(t')dt' \right|}{|z(t)|} \ll 1, \tag{15}$$

where $z_{\rm cl}(t)$ is the solution of Eq. (8a) for $N \to \infty$. To write the second form of the criterion of validity of the $1/N$-expansion we introduce following [6,7] the "convergence radius" $R = \left\{[\mathrm{Re}(\delta\alpha)]^2 + [\mathrm{Im}(\delta\alpha)]^2\right\}^{1/2}$. Then, the expansion is correct within a time interval when

$$\frac{R(t)}{|z(t)|} \simeq \frac{B^{1/2}(t)}{N^{1/2}|z(t)|} \ll 1. \tag{16}$$

As a rule, the both conditions, Eqs. (15) and (16), determine the same time interval for validity of the $1/N$-expansion [6,7] (For a physically interesting exception, the problem of SHG, see Sec. IV).

For dissipative systems with a simple attractor, the classical field intensity $|z_{\rm cl}(t)|^2$, as well as cumulants $\overline{B}(t)$, $C(t)$ and quantum correction $Q(t)$ are proportional to the factor $\exp(-\gamma t)$ and therefore, as follows from Eqs. (15) and (16) with account of Eq. (14), the $1/N$-expansion is well defined only in the time interval of order of several relaxation times: $t^* \simeq \gamma^{-1}$ [19]. Moreover, during this time interval the influence of quantum diffusion on the system dynamics is small.

A quite different behavior is characteristic for the stable limit cycle. Here, a variation near classical trajectory collapses to zero ($\delta\alpha \to 0$) and therefore $\overline{B}(t) \simeq |\delta\alpha|^2 \to 0$, $C(t) \simeq (\delta\alpha)^2 \to 0$. However $|z_{\rm cl}(t)| \simeq 1$ for the limit cycle and, as a result, the time interval of validity of the $1/N$-expansion is fairly large $t^* \simeq N\gamma^{-1}$. What is important, the diffusion is the major physical mechanism responsible for the difference between the classical and quantum dynamics for a stable limit cycle. In the two following sections we consider two typical examples of dissipative optical systems with a simple attractor and limit cycle.

III. NONCLASSICAL STATES GENERATION IN HIGHER-ORDER ANHARMONIC OSCILLATORS

We start by defining the squeezing and the Fano factor. Define the general field quadrature as $X_\theta = a\exp(-i\theta) + a^\dagger \exp(i\theta)$, where θ is the local oscillator phase. A state is said to be squeezed if there exists some value θ for which the variance of X_θ is smaller than the variance for a coherent state or the vacuum [1,9]. Minimizing the variance of X_θ over θ, we get the condition of so-called principal squeezing [1,9,10] in the form

$$S \equiv 1 + 2N(\langle|\delta\alpha|^2\rangle - |\langle(\delta\alpha)^2\rangle|) = 2(B - |C|) < 1. \tag{17}$$

The determination of the principal squeezing S is very useful because it gives the maximal squeezing measurable by the homodyne detection [1,9].

Another important characteristic of nonclassical properties of the light is the Fano factor $F = (\langle n^2\rangle - \langle n\rangle^2)/\langle n\rangle$, that determines the deviation of probability distribution from the Poissonian [1,9]. Substituting expression $a \to z + \delta\alpha$ into the expressions for $\langle n\rangle = N\langle a^\dagger a\rangle$ and $\langle n^2\rangle = N^2\langle a^\dagger a a^\dagger a\rangle = N^2\langle a^{\dagger 2} a^2\rangle + \langle n\rangle$ and after the Talor expansions in the first order of $1/N$, we have

$$F = 2B + \left(\frac{z^*}{z}C + \text{c.c.}\right). \tag{18}$$

We see, that in order to determine the time dependence of the principal squeezing S in (17) and the Fano factor (18) for nonlinear oscillators, we need to find the time dependence of z, C, and B from (8) for the Hamiltonian (1). Following the general procedure described in previous section, we first neglect quantum correction Q/N in Eq. (8a). In this case, equation (8a) has an exact solution in the form

$$z(t) = z_0 \exp\left[(-i\Delta - \gamma/2)t\right] \exp\left[-ig_l|z_0|^{2l}\mu_l(t)\right], \quad \mu_l(t) \equiv \left[1 - \exp(-\gamma l t)\right]/\gamma l. \tag{19}$$

We find the differentials dz and dz^* of classical solution (19), and using the substitution $|dz|^2 + \tilde{B} \to B$ and $(dz)^2 \to C$, we get

$$\begin{aligned} C(t) &= -l z_0^2 |z_0|^{2(l-1)} g_l \mu_l(t) \left(l|z_0|^{2l} g_l \mu_l(t) + i\right) \exp\left[(-\gamma - i2\Delta)t - i2|z_0|^{2l} g_l \mu_l(t)\right], \\ B(t) &= \exp(-\gamma t)\left[1/2 + l^2|z_0|^{4l} g_l^2 \mu_l^2(t)\right] + (\langle n_d\rangle + 1/2)\gamma t, \end{aligned} \tag{20}$$

where we have taken into account the initial conditions for B and C, Eq. (11). By substituting formulas (20) into Eq. (17), we obtain in the limits $\tau \equiv g_l(N)t \ll 1$ and $\gamma t \ll 1$ a very simple dependence of S on time as

$$S(t) = 1 - \left[l x_0^{2l} - (\gamma/g_l)\langle n_d\rangle\right] 2\tau < 1, \tag{21}$$

where for the sake of simplicity we have assumed that the initial value z_0 is real, $x_0 = \mathrm{Re} z_0$, and we have taken into account only terms that are linear in τ and γt. The short-time approximation $\tau \ll 1$ as well as the limit of a large photon number $N \gg 1$ are quite realistic for a nonlinear medium modelled by the anharmonic oscillators (for numerical estimates, see [1], chap. 10, and [10]). It should be noted that our formula (21) coincides with the corresponding formula for $S(t)$ in [10] for the Kerr nonlinearity ($l = 1$) without loss ($\gamma = 0$). In the case of no loss ($\gamma = 0$), our formula (21) shows that the rate of squeezing is determined by the factor $2l x_0^{2l}\lambda_l N^l \equiv 2l\mathcal{P}^{(2l+1)}$. Since λ_l is proportional to the $(2l+1)$-order nonlinear susceptibility, the factor $\mathcal{P}^{(2l+1)}$ has a physical meaning of nonlinear polarization. Therefore, the stronger is nonlinear the polarization induced by light in the medium, the more effective squeezing of light is possible. For a finite dissipation $\gamma \neq 0$, the squeezing is determined by an interplay between the polarization of nonlinear medium modelled by the anharmonic oscillator and the thermal fluctuations of the reservoir. As follows from (21), there exists the a critical number of phonons $\langle n_d\rangle^{(cr)} = (l/\gamma)\mathcal{P}^{(2l+1)}$ such that for $\langle n_d\rangle \geq \langle n_d\rangle^{(cr)}$ the squeezing is no longer possible.

In the same approximation, we obtain from (18) the following time dependence of the Fano factor

$$F(t) = 1 + 2\langle n_d\rangle\gamma t. \tag{22}$$

Thus, the statistic is super-Poissonian for any $\gamma \neq 0$ and is independent of the degree of nonlinearity l. This is in a good agreement with the earliest result of [9] for the case of a dissipative Kerr oscillator ($l = 1$), where the impossibility of sub-Poissonian statistics and antibunching were found from the exact solution.

Turn now to the discussion of the ranges for validity of our approach. It is easy to see that in terms of our approach the time dependence of the number of quanta for $l = 1$ is

$$\langle n\rangle(t)+1/2=N|z|^2+B\approx N|z_0|^2(1-\gamma t)+\langle n_d\rangle\gamma t,\quad \gamma t\ll 1,\quad g_l t\ll 1, \tag{23}$$

where we have used expressions (20) for cumulants B and C. It is instructive to compare (23) with the exact solution for $\langle n\rangle(t)$ for the Kerr nonlinearity [9]

$$\langle n\rangle(t)=\langle n_0\rangle\exp(-\gamma t)+[1-\exp(-\gamma t)]\langle n_d\rangle. \tag{24}$$

Eq. (23) and Eq. (24) both describe the evolution of an initially coherent state to a final chaotic state being characteristic for the reservoir. It is evident that formulas (24) and (23) coincide, when $\gamma t\ll 1$ and $\langle n_0\rangle\simeq N\gg 1$. A more accurate analysis of the condition for validity of the $1/N$-expansion should include a comparison of the solution of quantum motion equation (8a), which takes into account the quantum correction Q/N given by (9), with the solution of classical motion equation (19). It may be shown after some algebra, that if $\gamma t\ll 1$ and $\tau\ll 1$, the influence of the quantum correction Q/N on the dynamics of the mean value z is of the order $1/N$ and, therefore, our cumulant expansion is well-defined for $N\gg 1$. The same conclusion could be obtained considering another criterion of validity (16).

IV. QUANTUM-CLASSICAL CORRESPONDENCE IN SELF-PULSING REGIME OF SECOND-HARMONIC GENERATION

We now consider another example of a quantum optical system, namely intracavity SHG. The Hamiltonian describing two interacting quantum modes in the interaction picture has the form [11,14]

$$H=\sum_{j=1}^{2}\Delta_j b_j^\dagger b_j+iEN^{1/2}(b_1^\dagger-b_1)+\frac{i\chi}{2}(b_1^{\dagger 2}b_2-b_1^2b_2^\dagger), \tag{25}$$

where the boson operators b_j ($j=1,2$) describe fundamental and second-harmonic modes, respectively, Δ_j is the cavity detuning of mode j, $EN^{1/2}$ is the classical field driving first mode (E is of order of unity), χ is a second-order nonlinear susceptibility. The linear reservoir and its interaction with a second-order nonlinear medium are described by the Hamiltonians (2). Now we can rewrite full Hamiltonian of the problem in the form $H=N\mathcal{H}$, where $\mathcal{H}$ has the same form as (25) and (2) with account of the replacements analogous to (4) and definition of new coupling constant

$$g=\chi\sqrt{N}, \tag{26}$$

which is of order unity. Formally, the procedure of the $1/N$-expansion developed in sec. II can not be applied to the problem of SHG, however its straightforward generalization to the case of two interacting modes gives in the first order of $1/N$ the following self-consistent set of equations

$$\dot z_1=-\frac{\gamma_1}{2}z_1+E+gz_1^*z_2+\frac{1}{N}gB_{12}, \tag{27a}$$

$$\dot z_2=-\frac{\gamma_2}{2}z_2-\frac{g}{2}z_1^2-\frac{1}{N}\frac{g}{2}C_1, \tag{27b}$$

$$\dot B_1=-\gamma_1(B_1-B^{(0)})+gB_{12}^*z_1+gB_{12}z_1^*+C_1^*z_2+C_1z_2^*, \tag{27c}$$

$$\dot B_2=-\gamma_2(B_2-B^{(0)})-gB_{12}^*z_1-gB_{12}z_1^*, \tag{27d}$$

$$\dot C_1=-\gamma_1C_1+2g(C_{12}z_1^*+B_1z_2), \tag{27e}$$

$$\dot C_2=-\gamma_2C_2-2gC_{12}z_1, \tag{27f}$$

$$\dot C_{12}=-0.5(\gamma_1+\gamma_2)C_{12}+gB_{12}z_2-C_1z_1+C_2z_1^*, \tag{27g}$$

$$\dot{B}_{12} = -0.5(\gamma_1 + \gamma_2)B_{12} + gC_{12}z_2^* + gz_1(B_2 - B_1), \tag{27h}$$

where $z_j \equiv \langle a_j \rangle = N^{1/2}\langle b_j \rangle$, $B_j = N\langle \delta\alpha_j^* \delta\alpha_j \rangle + 0.5$, $C_j = N\langle (\delta\alpha_j)^2 \rangle$ $(j = 1, 2)$, $B_{12} = N\langle \delta\alpha_1^* \delta\alpha_2 \rangle$, $C_{12} = N\langle \delta\alpha_1 \delta\alpha_2 \rangle$ and $B^{(0)}$ is defined in Eq. (12). The initial conditions for system (27) are $B_j(0) = 1/2$, $C_j(0) = C_{12}(0) = B_{12}(0) = 0$, $z_2(0) = 0$, and $z_1(0) = z_0$, where z_0 is of order unity. In this work, we limit ourselves only by the values of the field strength z_0 corresponding to self-oscillations [11] and $\Delta_1 = \Delta_2 = 0$.

It is easy to see that in the limit $N \to \infty$ and for $g = \text{const} \simeq 1$, we get from Eqs. (27a) and (27b) correct classical motion equations for the scaled field amplitudes. The solution of motion equations (27c)-(27h) for the second-order cumulants has the form

$$\mathbf{X}(t) = \overline{\mathbf{X}}(t) + \left(\gamma B^{(0)}t, \gamma B^{(0)}t, 0, 0, 0, 0\right), \quad \mathbf{X}(t) \equiv [B_1(t), B_2(t), C_1(t), C_2(t), B_{12}(t), C_{12}(t)], \tag{28}$$

where vector $\overline{\mathbf{X}}$ describes the part of $\mathbf{X}$ that can be obtained by linearization around a classical trajectory. Variations near a stable limit cycle rapidly approach zero and therefore $\overline{\mathbf{X}}(t) \to 0$. As a result, we have only a diffusive growth of cumulants B_j $(j = 1, 2)$ as

$$B_j(t) = 0.5\gamma_j t, \tag{29}$$

where we considered the case of a quiet reservoir $\langle n_d \rangle$. This result indicates that the influence of reservoir zero-point energy on the dynamics of the nonlinear system is principal physical mechanism responsible for the difference between the classical and quantum dynamics in the semiclassical limit. A time scale t^* for a correct description of the dynamics of quantized SHG in terms of classical electrodynamics can be found by using criterion (16). Taking into account that $|z(t)| \simeq 1$, we have $t^* \simeq 2N\gamma^{-1}$.

Note that the quantum corrections to the classical motion equations (27a) and (27b)) do not include cumulants $B_{1,2}$. Therefore, in the first order of $1/N$, there is no difference between the evolution of quantum mean values and the classical dynamics for limit cycle. In other words, the quantum correction $Q \to 0$, and therefore criterion (15) of validity of $1/N$-expansion does not work. In this respect, the quantized SHG is a somewhat singular problem. In other quantum optical systems, for instance, for a nonlinear oscillator with $l \geq 1$, typically both criteria of validity (16) and (15) give the same result.

Over a decade ago, Savage addressed the same problem of quantum-classical correspondence at self-oscillations in SHG numerically [14]. He calculated Q distribution function in the Gaussian approximation centered on a deterministic trajectory corresponding to a limit cycle. He worked at a large field and small nonlinearity limits, $\chi/\gamma_{1,2} \to 0$, which correspond to the classical limit [14]. It is easy to see that the condition $\chi/\gamma_{1,2} \to 0$ is consistent with our condition $N \gg 1$, if one additionally considers the natural condition of a not very strong dissipation in Eqs. (27), $\gamma_{1,2}/g \lesssim 1$ together with $g \simeq 1$ [Eq. (26)]. In other words, Savage's small parameter χ/γ corresponds to our large parameter N as $\chi/\gamma \to N^{-1/2}$. To establish the difference between the classical and quantum dynamics, the motion equations for law-order cumulants were obtained in [14] and solved numerically for particular values of the parameters [21]. Based on the results of numerical simulations, Savage concluded that it was a quantum diffusion that was mostly responsible for the difference between the classical and quantum dynamics in the semiclassical limit. Moreover, his numerical estimate for a characteristic time for the classical description scales as $(\gamma/\chi)^2$, which is in a good agreement with our analytical result $t^* = 2\gamma^{-1}N$. In summary, our analytical results for the quantum-classical correspondence at self-pulsing in SHG are quite consistent with the previous numerical investigation of same problem in [14].

V. CONCLUSION

We developed the method of $1/N$-expansion to consider the nonlinear dynamics and nonclassical properties of light in dissipative optical systems in the limit of a large number of photons. The method was applied to the investigation of squeezing in higher-order dissipative nonlinear oscillators. We would like to note that our method can also be directly applied to an important case of nonclassical states generation in a medium involving competing nonlinearities [22].

We found a time scale of validity of the $1/N$-expansion for a classical description of the dynamics of nonlinear optical systems with a simple attractor and a limit cycle. For systems with a simple attractor, this time scale is of order unity, and for a limit cycle – proportional to a large N. Qualitatively this result can be understood as follows. For time of order unity, the trajectory spirals around a stable stationary point with a small amplitude and therefore in virtue of the uncertainty principle, the contribution of quantum corrections to the classical motion equations becomes

very important. Unlike the previous case, the oscillations corresponding to a limit cycle are often close to harmonic and thus their quantum and classical descriptions can coincide for a fairly long period of time. The basic difference between the classical and quantum dynamics in the latter case originates from the influence of reservoir zero-point fluctuations, which in our notations are of order of $1/N$. This result is in a good agreement with the result of earliest numerical simulations of self-oscillations in the quantized second harmonic generation [14]. Finally, it should be noted that our findings are of a rather general nature and can be applied to the investigations of self-oscillations in other optical systems, for example, in the optical bistability [23–25].

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* E-mail: kna@tnp.krascience.rssi.ru
** E-mail: perina@optnw.upol.cz

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